Basic differentiation

Differentiation is defined as *the rate of change of a function.* It is used so that we can solve gradient problems, maximum and minimum points of graphs, modelling problems, and kinetic problems. These are all explained in more detail below.

The notation for a differentiated function is either $\frac{dy}{dx}$ (if the equation is in the form $y =$) or f' if the function is given as f(x).

How do you differentiate?
The simple answer to this is to reduce the index (the power) by 1 and multiply by the old power. Below are some examples of some simple differentiation.

Function	Differential
$y = x^2 + 3x + 5$	$\frac{dy}{dx} = 2x + 3$
$f(x) = 3x^4 + 5x^3$	$f' = 12x^3 + 15x^2$

It is useful to remember some of your simple index facts.

Function	Can be written as:
$\sqrt{x}$	$x^{\frac{1}{2}}$
$\sqrt[3]{x}$	$x^{\frac{1}{3}}$
$\frac{1}{x}$	x^{-1}
$\frac{1}{\sqrt{x}}$	$x^{-\frac{1}{2}}$

So these can be differentiated as shown below

Function	Differential
$y = \sqrt{x} = x^{\frac{1}{2}}$	$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
$f(x) = \frac{1}{3x} = \frac{x^{-1}}{3}$	$f' = \frac{-2x^{-2}}{3} = \frac{-2}{3x^2}$

The ln, e, and trig functions have set differentials and are given in the formula sheet as follows:

Function	Differential
sin x	cos x
cos x	- sin x
e^x	e^x
ln (x)	$\frac{1}{x}$

The Chain rule

The chain rule is used we have composite functions, or one function inside another, such as those shown below:

$y = \sin(3x^2)$ $y = e^{5x+2}$ $y = \ln(\sqrt{x})$ $y = (x^2 + 3)^7$

The basic rule for the chain rule is: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

Examples:

1. $y = \sin(3x^2)$

$y = \sin u$ $\frac{dy}{du} = \cos u$

$u = 3x^2$ $\frac{du}{dx} = 6x$

$$\frac{dy}{dx} = 6x\cos u = 6x\cos(3x^2)$$

2. $y = e^{5x+2}$

$y = e^u$ $\frac{dy}{du} = e^u$

u = $5x + 2$ $\frac{du}{dx} = 5$

$$\frac{dy}{dx} = 5e^u = 5e^{5x+2}$$

3. $y = \ln(\sqrt{x})$

$y = \ln(u)$ $\frac{dy}{du} = \frac{1}{u}$

$u = \sqrt{x}$ $\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}}$

$$\frac{dy}{dx} = \frac{1}{u} \times \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2x}$$

4. $y = (x^2 + 3)^7$

$y = u^7$ $\frac{dy}{du} = 7u^6$

$u = x^2 + 3$ $\frac{du}{dx} = 2x$

$$\frac{dy}{dx} = 7u^6 \times 2x = 14x(x^2 + 3)$$

Product and Quotient Rules

When you have two functions of x multiplied by each other you use the product rule.

When you have one function of x divided by another you use the quotient rule.

The formulae are in the IB information booklet.

Product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient rule: $\frac{d}{du}(\frac{u}{v}) = \frac{u\frac{dv}{dx} + v\frac{du}{dx}}{v^2}$

Example of Product rule

Find $\frac{dy}{dx}$ if $y = x^2 \cos x$.

$u = x^2 \qquad \frac{du}{dx} = 2x$

$v = \cos x \qquad \frac{dv}{dx} = -\sin x$

$\frac{dy}{dx} = x^2(-\sin x) + 2x(\cos x)$

Example of the quotient rule

Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{5x}$

$u = \sin x \qquad \frac{du}{dx} = \cos x$

$v = 5x \qquad \frac{dv}{dx} = 5 \qquad v = 25x^2$

$\frac{dy}{dx} = \frac{5x\cos x - 5\sin x}{25x^2}$

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$$\frac{dy}{dx} = \frac{x\cos x - \sin x}{5x^2}$$

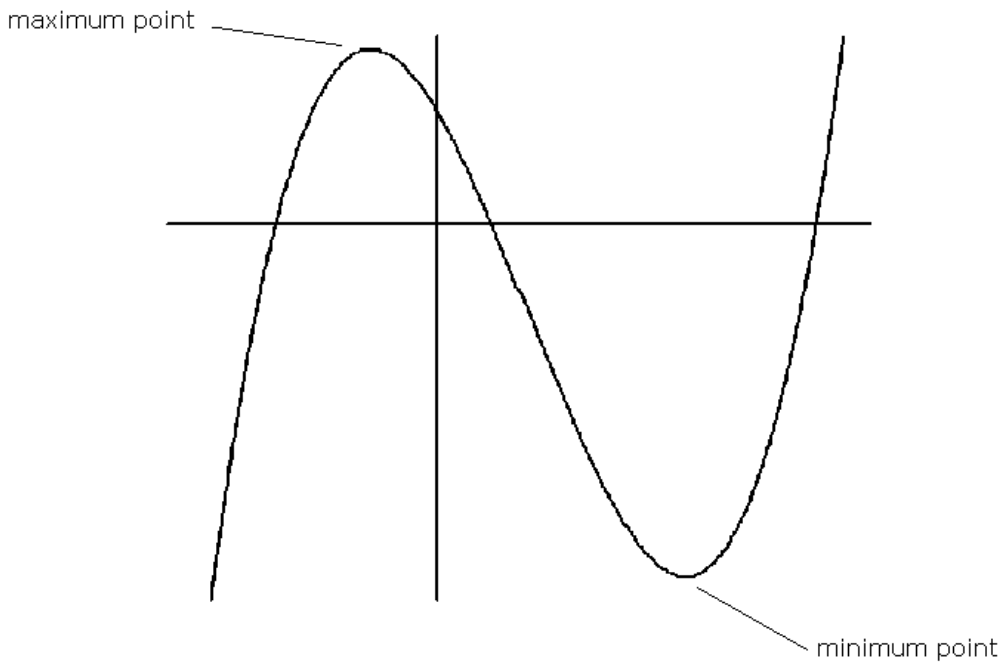
Applications

1. The gradient function

The first application of differentiation is to find the gradient of a function. The $\frac{dy}{dx}$ is the gradient of the function.

2. Maximum and minimum points

Curves can have maximum and minimum points, as shown in the diagram below.



At the curve's maximum or minimum point the gradient = 0. This will help to calculate the *x*-coordinate when a curve is a maximum or minimum.

By differentiating again (shown as $\frac{d^2y}{dx^2}$) and substituting the value of x into the $\frac{d^2y}{dx^2}$ function we can determine whether the point is a maximum or a minimum.

+ve value = minimum *-ve value = maximum*

3. Equations of tangents

A tangent is a line that touches the outside of a curve at a given point. It will always be a straight line in the form $y = mx + c$.

We can find equations of tangents by first finding the gradient (m) from the differentiated function and then finding the coordinate at the point where the tangent meets the curve and substituting back into the $y = mx + c$ equation. An example of this is shown below in the guided example.

Guided example

The function $f(x) = x^3 + 3x^2 - 10x$

(a) Find the gradient when $x = 2$.

(b) Find the x-value of both the maximum and minimum points and distinguish between them.

(c) Find the equation of the tangent to the curve at the point where $x = 2$

Answer (a)

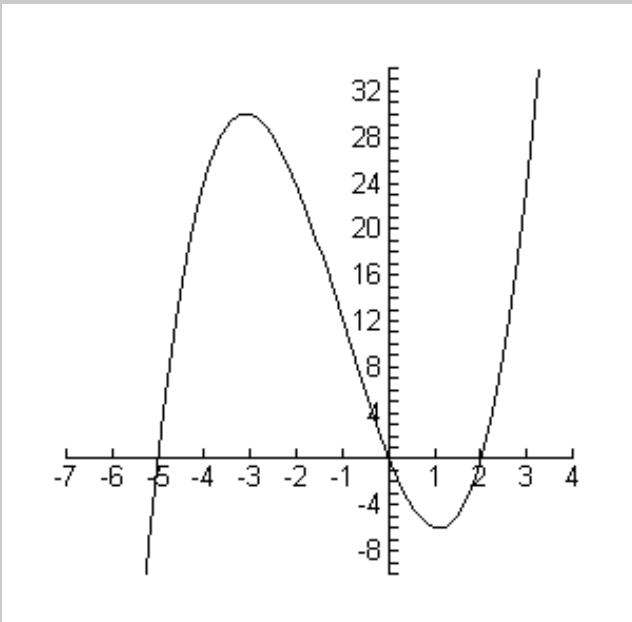
Differentiate the function and substitute the value of x given in the question as shown below.

$f'(x) = 3x^2 + 6x - 10$

$x = 2 \quad 3(2)^2 + 12 - 10 = 14$.

Answer (b)

First off draw a graph on your GDC to get some idea of where the maximum and minimum points are.



Now take the differentiated equation and make the equation equal to zero. This is because the gradients at the maximum and minimum points have a zero gradient.

We will need the quadratic formula to solve this.

$3x^2 + 6x - 10 = 0$

$$\frac{-6 \pm \sqrt{36+120}}{6} = \frac{-6 \pm 12.5}{6}$$

$x = 1.08$ and -3.08

To distinguish between them will require differentiating again, although we can clearly see from the graph that 1.08 is the minimum and 3.08 is the maximum.

$f'' = 6x + 6$.

By substituting in 1.08 we get a +ve number, therefore a minimum point.
By substituting -3.08 we get a –ve number, therefore a maximum point.

Answer (c)

From the question above we already have the gradient of the tangent = 14/
So the equation of the tangent is $y = 14x + c$.

At the point where $x = 2$, $y = 0$, so we have the equation $0 = 28 + c$.
So c is equal to –28 and the equation of the tangent is:

$y = 14x - 28$.

Basic integration

Integration is the opposite of differentiation. The mathematical sign for integration is:

$\int f(x)dx$, where $f(x)$ is a function of x and you are integrating with respect to x.

To integrate simply add 1 to the index and divide by the new power. Unless you are finding a definite integral (see below), you must always add on a constant, c.

Below are some simple integrals.

Function	Integral
$\int(x^2 + 4x^3)dx$	$\frac{x^3}{3} + x^4 + c$
$\int(\frac{1}{x^2} + \sqrt{x})dx$	$-\frac{1}{x} + \frac{2x^{\frac{3}{2}}}{3} + c$

Just like differentiating there are some special integrals that are given in the formulae sheet, these are summarised below:

Function	Integral
sin x	- cos x + c
cos x	sin x + c
e^x	e^x + c
$\frac{1}{x}$	ln (x) + c

Composite functions

When differentiating we composite functions we used the chain rule (see above). For integrating we use a similar method. Two examples with explanations are shown below.

1. Choose a value for *u*. This will usually be the inner function.

2. Rewrite the equation with the substitution. At this point we cannot integrate because we have *dx* and not *du*.

3. To find a value for *dx* we differentiate *u* with respect to *x* ($\frac{du}{dx}$) and rearrange the equation to make *dx* the subject.

4. Now substitute *dx* with your equation. Remove any constants outside the integral sign.

5. Integrate with respect to *u* and finally substitute *x* back into the equation.

Example 1

$\int (3x+7)^5 dx$

$u = 3x + 7$

$\int u^5 dx$

$\frac{du}{dx} = 3$, so $dx = 3du$.

$\int u^5 \times 3du = 3\int u^5 du$

$= \frac{3u^6}{5} + c = \frac{3(3x+7)^6}{5} + c$

Example 2

$\int (4\cos(5x+3))dx$

$u = 5x + 7$

$4\int \cos(u)dx$

$\frac{du}{dx} = 5$, so $dx = \frac{du}{5}$

$\frac{4}{5}\int \cos(u)du$

$= \frac{4}{5}\sin(u) + c = \frac{4}{5}\sin(5x+7) + c$

The value of *c*

In some situations the value of *c* can be found if there is extra information in the question, such as a value of $f(x)$ and its corresponding x value.

For example:
$f'(x) = 3x^2+x+1$. Find $f(x)$ if $f(2)=15$

First integrate to find $f(x)$: $f(x) = x^3 + \frac{x^2}{2} + x + c$

Now substitute in the values: $15 = 2^3 + \frac{2^2}{2} + 2 + c$ $c = 3$

Final answer is: $f(x) = 2^3 + \frac{2^2}{2} + 2 + 3$

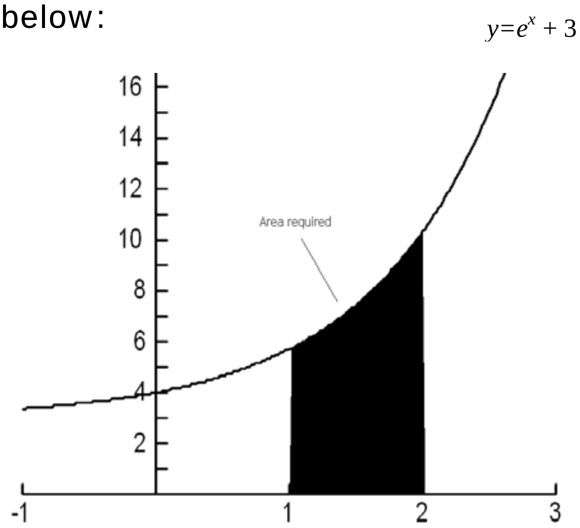
Definite integration

Definite integration is when we have values on the integration sign. This is used to find the area under a graph. When we integrate definitely we no longer have the value of *c*.

For example:

$\int_1^2 (e^x + 3)dx$ is asking to find the value between the curve and the *x*-axis between $x=1$ and $x=2$.

This is shown graphically below:



To solve numerically:

$\int_1^2 (e^x + 3)dx = \left[e^x + 3x\right]_1^2$ (note the use of square brackets and the numbers at the end)

Now we simply substitute the higher number in first and take away the smaller number when substituted.

$(e^2 + 6) - (e^1 + 3) =$ 7.67 units^2.

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Kinetics
Kinetics is about motion, distance travelled, velocity (speed with direction), and acceleration. Here are some simple definitions:

Velocity is the change in distance with respect to time.	The distance equation differentiated with respect to time.
Acceleration is the change in velocity with respect to time.	The velocity equation differentiated with respect to time.

Calculus is often used to help solve these motion problems when equations are given in the form with time as the variable.

In these types of question you may be given any type of the three functions for distance, velocity, or acceleration. The variable for each will be time.

Distance function.
Differentiate once to get the velocity function.
Differentiate again to get the acceleration function.

Velocity function.
Integrate to get the distance function – remember that the area under a velocity-time curve gives the distance travelled.
Differentiate to get the acceleration function.

Acceleration function.
Integrate once to get the velocity function.
Integrate again to get the distance function.

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Guided example

A rocket is launched in the air. Its velocity at time is given by the function,

$v = 80 - 12t$ where v is a measure in m/s and t is in seconds.

(a) Find an expression for the height of the rocket, that is its displacement from the ground at time t. You may assume that the rocket starts from 0 metres.
(b) Find the maximum height of the rocket in metres
(c) Find the constant acceleration of the rocket in m/s^{-2}.

Answer (a)

As we are given the velocity, we know that differentiating will give the acceleration and integrating will give the displacement, or height of the rocket.

$\int (80-12t)dt = 80t - 6t^2 + c$ Because height = 0 when t = 0, c = 0.

$h = 80t - 6t^2$

Answer (b)

The maximum height can be found by differentiating and making the function equal 0. As the differential of the displacement is the velocity, we have:

$80 - 12t = 0$
$t = 6\frac{2}{3}$.
Substituting that back into the equation for height he have:
$h = 80(6\frac{2}{3}) - 6(6\frac{2}{3})^2$
h = 266 metres.

Answer (c)

The acceleration can be found by differentiating the velocity.

$v = 80 - 12t$

$\frac{dv}{dt} = -12$

So constant acceleration is –12 $m/s-^{2}$.