Basics of Matrices

Matrices can be: added together or subtracted (if the matrices are the same dimensions); multiplied by a scalar; multiplied together.

Addition and subtraction

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+b & b+f \\ c+g & d+h \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-b & b-f \\ c-g & d-h \end{bmatrix}$$

Multiplication by a scalar.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 5 & -2 \\ 0 & 6 & -3 \end{bmatrix} \qquad 4A = \begin{bmatrix} 8 & 16 & -4 \\ 12 & 20 & -8 \\ 0 & 24 & -12 \end{bmatrix}$$

Multiplying 2x2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} (ae+bg) & (af+bh) \\ (ce+dg) & (cf+dh) \end{bmatrix}$$

Multiplying 3x3 matrices

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \times \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} (aj+bm+cp) & (ak+bn+cq) & (al+bo+cr) \\ (dj+em+fp) & (dk+en+fq) & (dl+eo+fr) \\ (gj+hm+ip) & (gk+hn+iq) & (gl+ho+jr) \end{bmatrix}$$

Determinants

2x2 determinants:

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|x| = ad - bc$$

3x3 determinants:

$$y = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|y| = a(ei - fh) - b(di - fg) + c(dh - eg)$$

This can be remembered by taking the first line (*a, b, c*) and crossing out each corresponding line and column leaving a 2x2 matrix. Find the determinant of each 2x2 matrix and then multiply by the corresponding letter. Better explained diagrammatically:

$$\begin{bmatrix} \not{a} & \not{b} & \not{c} \\ \not{d} & e & f \\ \not{g} & h & i \end{bmatrix} = a(ei - fh)$$

$$\begin{bmatrix} \not{a} & \not{b} & \not{c} \\ d & \not{e} & f \\ g & \not{h} & i \end{bmatrix} = -b(di - fg)$$

$$\begin{bmatrix} \not{a} & \not{b} & \not{c} \\ d & e & \not{f} \\ g & h & \not{i} \end{bmatrix} = c(dh - eg)$$

The singular matrix.

The singular matrix is one whose determinant is known to be 0.
This is a common exam question.

Example

$$Y = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -1 \\ -4 & k & 3 \end{bmatrix}$$

Find the value of *k* such that matrix *Y* is singular.

2(k) – 3(3-4) -1(k) = 0
2k – 9 + 12 – k = 0
k = -3

Inverses

2x2 matrices

To find the inverse of a 2x2 matrix, follow the steps:

1. Find the determinant.
2. Swap the positions of *a* and *d*, and invert the signs of *b* and *c*.
3. Divide each element of the matrix by the determinant.

Example:

$$x = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$$

1. $|x| = 5 - 6 = -1$
2. $\begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}$
3. $\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$

An important aspect of matrices and their inverses is that they multiply together to give the identity matrix:

$\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, where the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix.

Using a Graphics Display Calculator (GDC)

Your GDC can evaluate inverses of 2x2 and 3x3 matrices (indeed any order higher than that). It is important that you become familiar with using your calculator to perform matrix calculations, and in particular to find the inverse of a 3x3 matrix.

Applications – Solving simultaneous equations

Using the properties of inverses in 3x3 matrices we are able to solve simultaneous equations with 3 unknowns as follows:

Consider the simultaneous equations:

$$3x - y + 2z = 20$$
$$2x + 2y - z = -1$$
$$5x + 3y + 4z = 32$$

From these equations we can form the matrix equation:

$$\begin{bmatrix} 3 & -1 & 2 \\ 2 & 2 & -1 \\ 5 & 3 & 4 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ -1 \\ 32 \end{bmatrix}$$

Performing the matrix multiplication to obtain the 3 simultaneous equations can check this.

Multiply each side by the inverse of the 3x3 matrix. You can obtain the inverse by using a GDC.

By doing this we are left with the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{11}{38} & \frac{5}{19} & \frac{-3}{38} \\ \frac{-13}{38} & \frac{1}{19} & \frac{7}{38} \\ \frac{-2}{19} & \frac{-7}{19} & \frac{4}{19} \end{bmatrix} \times \begin{bmatrix} 20 \\ -1 \\ 32 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

So x=3, y=-1, z=5.

Note: The multiplication can be performed using a GDC.