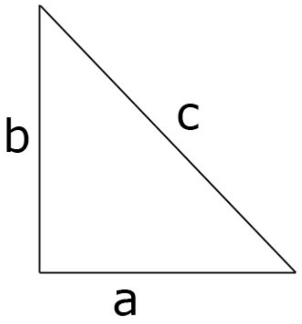
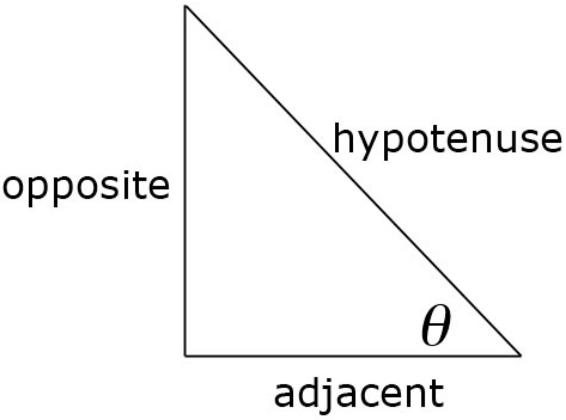
Right-angled triangles

Pythagoras' Theorem
You need to be able to calculate missing sides in right angled triangles, given two sides by using Pythagoras' theorem.



$$c^2 = a^2 + b^2$$
$$b^2 = c^2 - a^2$$
$$a^2 = c^2 - b^2$$

SOHCAHTOA
You need to find missing angles and missing sides in right angled triangles by use SOHCAHTOA (**S**in **O**pposite **H**ypotenuse **C**osine **A**djacent **H**ypotenuse **T**angent **O**pposite **A**djacent)



$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{o}{h}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{a}{h}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{o}{a}$$

Angles from 0° to 360°

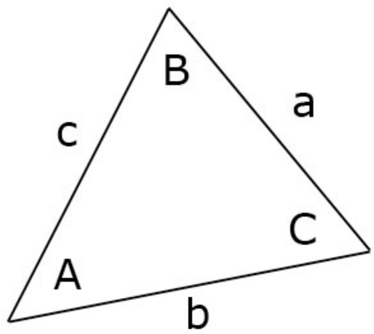
The calculator will give you one answer, usually between 0° and 90° for an equation in the form sin θ = x. However there will be two answers between 0° and 360°.

A basic rule is this:

• Calculate one value using your calculator	sin θ = 0.5 Type in *sin^{-1}(0.5)* = 30° cos θ = 0.7 Type in *cos^{-1}(0.7)* = 46° tan θ = 1.2 Type in *tan^{-1}(1.2)* = 30°
• For sin then the second answer will be 180 - θ	180 – 30 = 150 Answers are 30° and 150°
• For cos then the second answer will be 360 - θ	360 – 46 = 316 Answers are 46° and 316°
• For tan then the second answer will be 180 + θ	180 + 50 = 230 Answers are 50° and 230°

Non-right angled triangles

Formulae for non-right angled triangles are based on the diagram and notation below.



You should be able to apply the sine rule, cosine rule, and the area of a triangle. All of these formulae are in the IB formula booklet.

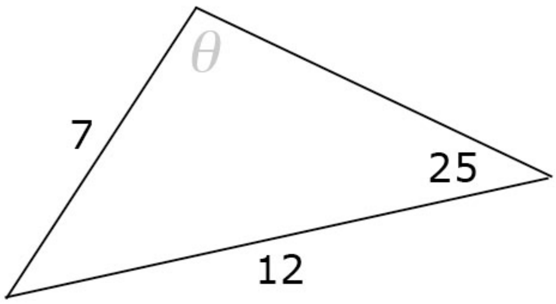
The sine rule (non-right angled triangle)

The sine rule is used when:

- You are given two sides and an angle you are missing an angle.
- You are given two angles and a side and you are missing a side.

Missing side:-	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
Missing angle:-	$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

Example
The missing angle in the triangle below is known to be obtuse. Find the missing angle.



Use the formula $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where a = 12, b = 7 and B = 25°.

The solution will be: $\frac{\sin A}{12} = \frac{\sin 25}{7}$

$$\sin A = \frac{\sin 25 \times 12}{7}$$

$\sin A =$ 0.724

$A = \sin^{-1}(0.724)$

A = 46°

But as A is obtuse (between 90° and 180°) the answer will be 180 – 46 = 134°.

The cosine rule (non-right angled triangle)

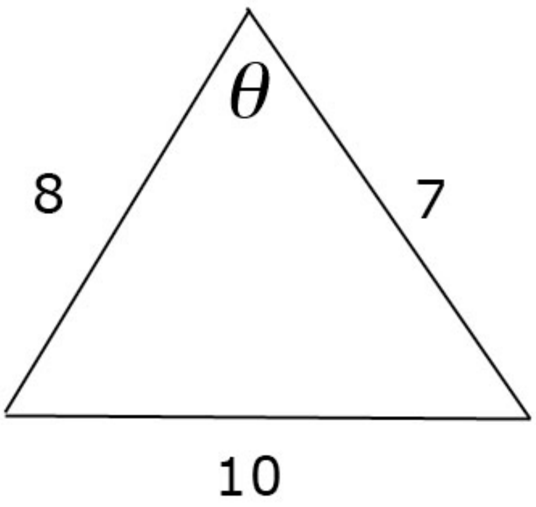
The cosine rule is used when:

- You are given three sides and you are missing an angle.
- You are given two sides and the angle opposite the missing side.

Missing side:-	$a^2 = b^2 + c^2 - (2bc\cos A)$
Missing angle:-	$\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Example

Find the missing angle in the triangle below.



Use the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, where *a* = 10 as it is opposite θ, *b* = 8 and *c* = 7.

The solution will be: $\cos\theta = \dfrac{7^2 + 8^2 - 10^2}{2 \times 7 \times 8}$

$\cos\theta = \dfrac{13}{112}$

$\theta = \cos^{-1}(\dfrac{13}{112})$

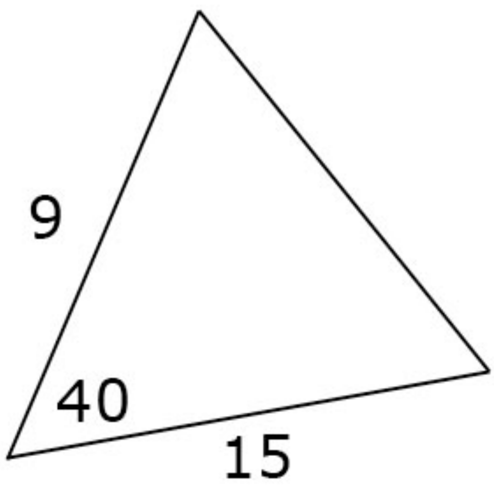
$\theta = 83^{\circ}$

Area of a triangle (non-right angled triangle)

The area of a triangle can be found if two sides are known and the angle between them is give. The formula is given below.

Area of a triangle:-	$\frac{1}{2}ab\sin C$

Example
Find the area of the triangle below.



Use the formula $\frac{1}{2}ab\sin C$, where a = 9, b = 15 and C = 40°.

$\frac{1}{2}$x15x9xsin(40)

= 43.4 units^2

Bearings

Bearings are always 3 figure angles. The basic rules for finding a bearing are:

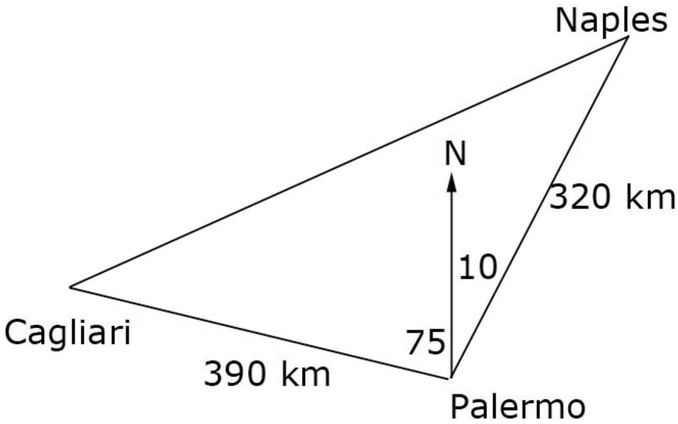
- Locate where you 'are'. This will be the location that follows *from*. E.g. If you are asked to find the bearing of Bangkok from Columbo, you will be at Columbo.
- Draw a line between the two locations.
- Draw a line going north from where you are.
- Draw an arc starting at the line going north and stopping when you reach the destination line.
- The angle of the arc is the bearing you need. Always give 3 figures in your final answer, so if the angle measure 35º, the bearing will be 035º.

Guided example

Two ships set sail from the port of Palermo. One sails to Cagliari a distance of 390 km on a bearing of 285º. The other ship sails to Naples on a bearing of 010º and a distance of 320 km.

(a) Draw a diagram to show the information given above.
(b) Use you diagram to find the area in km^2 between the towns of Palermo, Cagliari, and Naples.
(c) Find the bearing and distance from Cagliari to Naples.

Answer (a)



Answer (b)

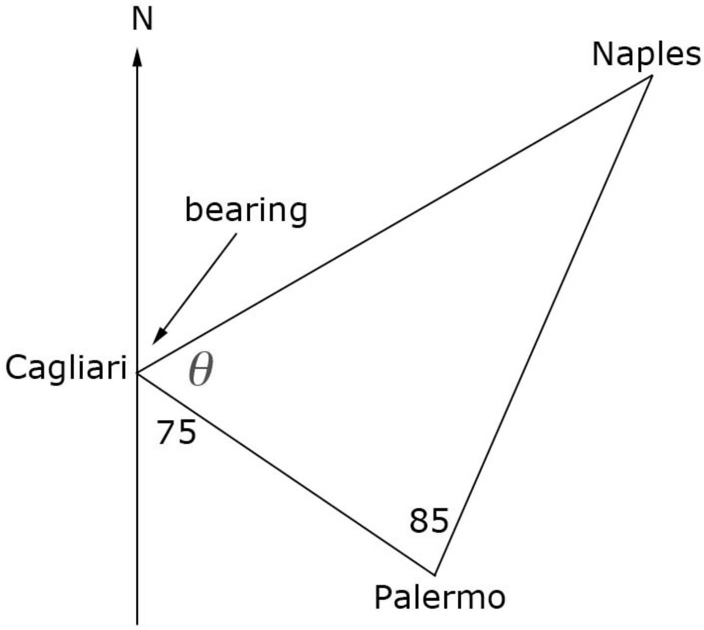
As the triangle is not a right-angled triangle then the $\frac{1}{2}abSinC$ formula:

$\frac{1}{2}\times 390\times 320\times \sin(85)$ = 62163 km^2

Answer (c)

The direct distance between C and N can be calculated using the cosine rule:

$CN^2 = 320^2 + 390^2 - (2x320x390xcos85)$
$= 232745$
CN = 482 km



The bearing is slightly harder. Look at the diagram below and the bearing is marked. The angle 75º is calculated using the fact that the two lines north (from P and from C) are parallel and therefore alternate angles equal.
Calculating θ will adding this with 75, before taking it away from 180 will give us the bearing.

The angle θ can be calculated using the sine rule:

$$\frac{\sin\theta}{320} = \frac{\sin 85}{482}$$

$$\sin\theta = 0.661$$
$$\theta = 41$$

The bearing is therefore 180 – (75 +41) = 064º

Radian measure

Radians are an alternative measure to degrees to measure the size of an angle.

A simple conversion is π radians = 180°.

This table shows some other degrees converted into radian measure:

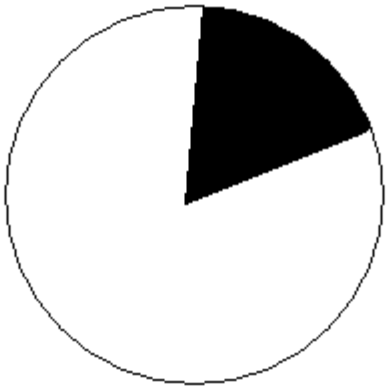
Degrees	Radians
30°	$\frac{\pi}{6}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
180°	π
270°	$\frac{3\pi}{2}$
360°	2π

Radians are often used to find arc lengths and areas of sectors.

Arc length = $r\theta$ and the sector area = $\frac{r^2\theta}{2}$, where r = radius and θ is the angle measured in radians.

Example

In the diagram opposite the radius = is 12cm and the angle of the minor sector at the centre of the circle is 0.8π radians.



a) Find the minor arc length.
b) Find the major sector area (not shaded in the diagram below).

a) $12 \times 0.8\pi = 30.16$ *cm*
b) *angle is* $2\pi - 0.8\pi = 1.2\pi$.

$$= \frac{12^2 \times 1.2\pi}{2} = 271.5cm^2$$

Sine and Cosine functions

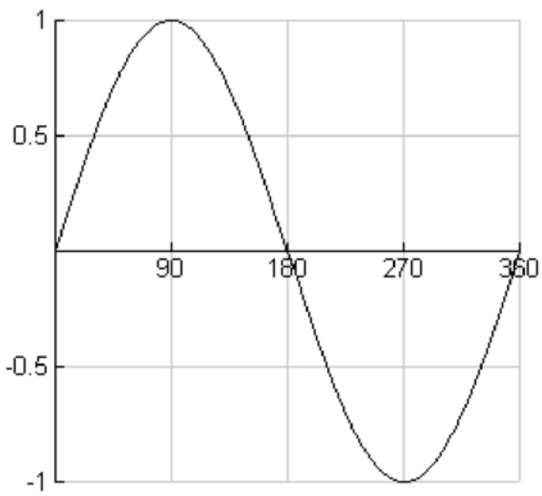
Sine and cosine curves are often used to model real life situations, such as hours of sun or tidal times. The basic sine and cosine curves are shown below.
It is important to have a basic grasp of the nature of these curve and trigonometry, especially angles between 0º and 360º. Also you should be aware of the value of sin and cos that give 0 and 1. These are shown below in the table.

Sin 0	Sin 90	Sin 180	Sin 360	Cos 0	Cos 90	Cos 270	Cos 360
0	1	0	0	1	0	0	1

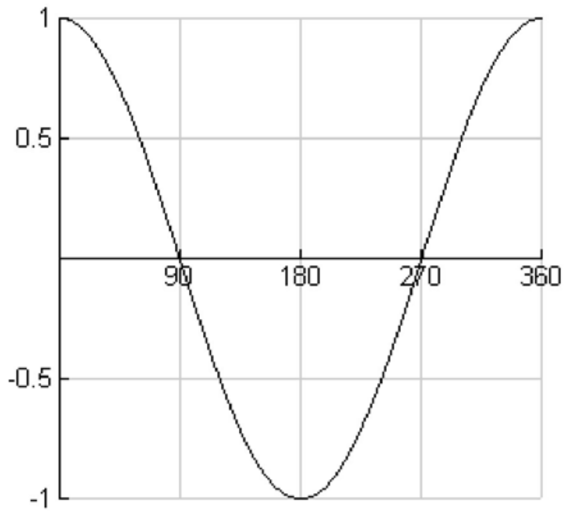
These values can be seen clearly from the sin and cos curves below.

Basic curves

y = sin x

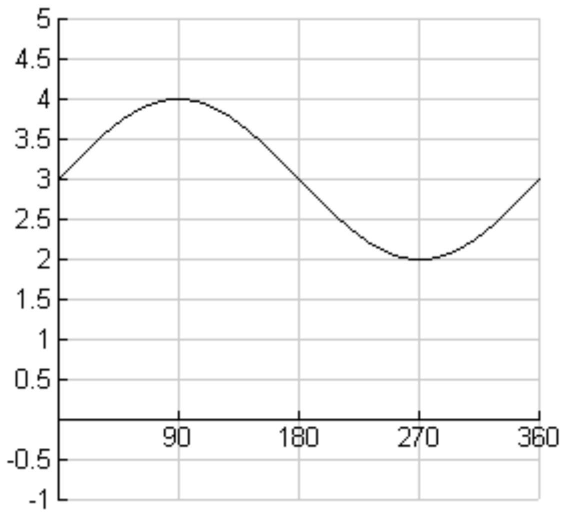


y = cos x

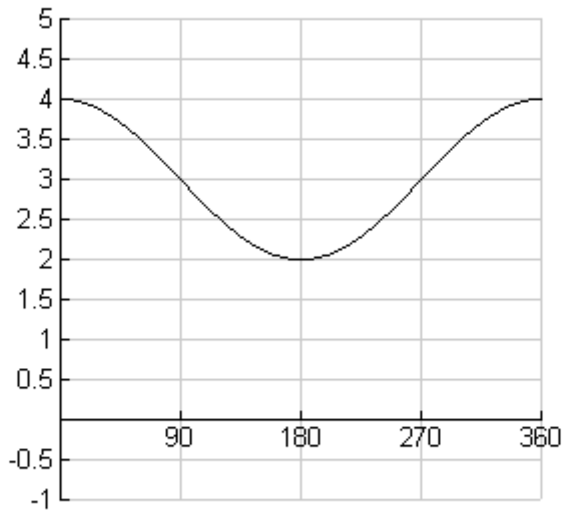


Adding a number to the curve

$y = (\sin x) + 3$



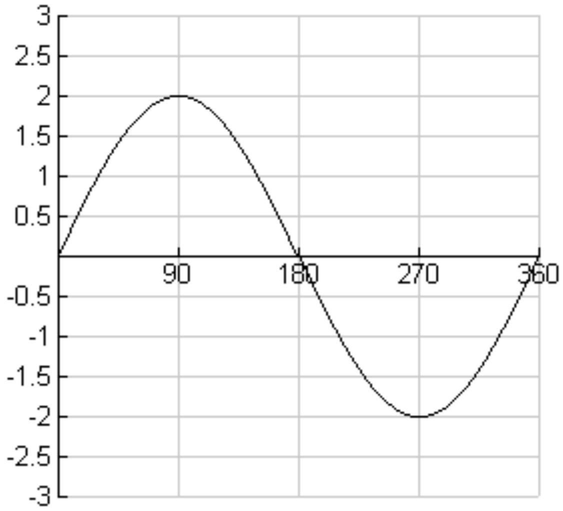
$y = (\cos x) + 3$



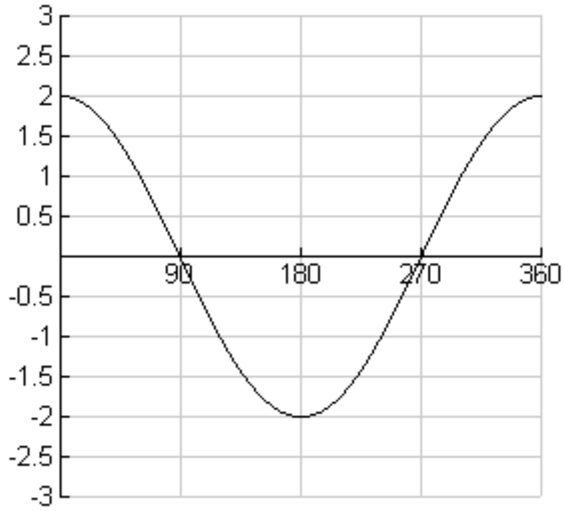
Adding a number to the curve causes the curve to translate (slide) up.
Subtracting a number from the curve causes the curve to translate (slide) down.

Multiplying by a number ***(the amplitude)***

$y = 2\sin x$



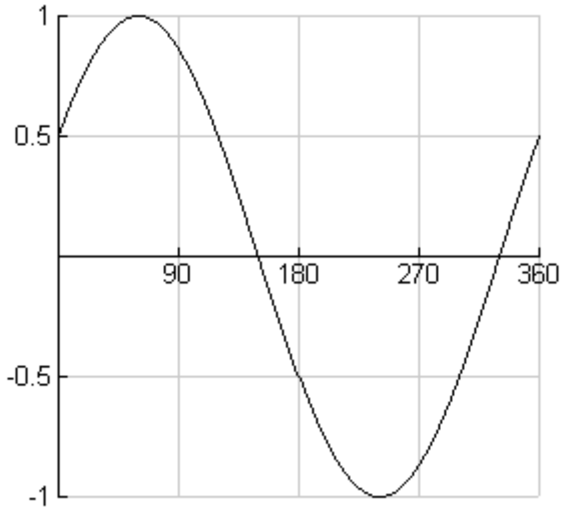
$y = 2\cos x$



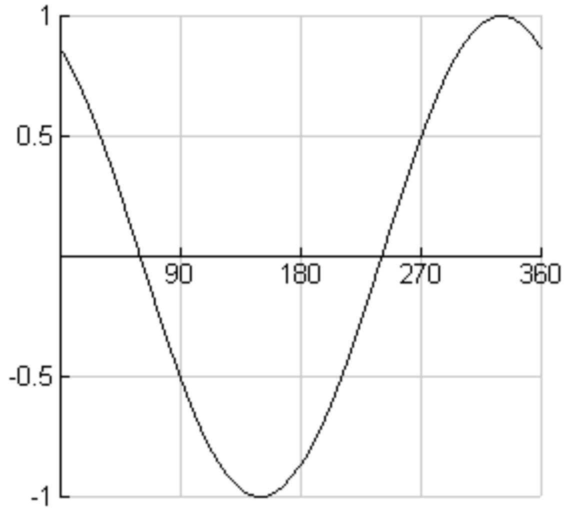
Multiplying the curve by a number causes the curve to be stretched. Each curve now has an **amplitude** of 2.

Adding a number to x

$y = \sin(x+30)$



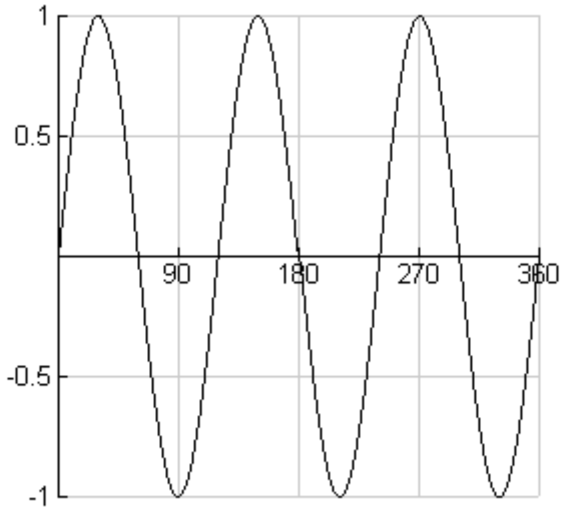
$y = \cos(x+30)$



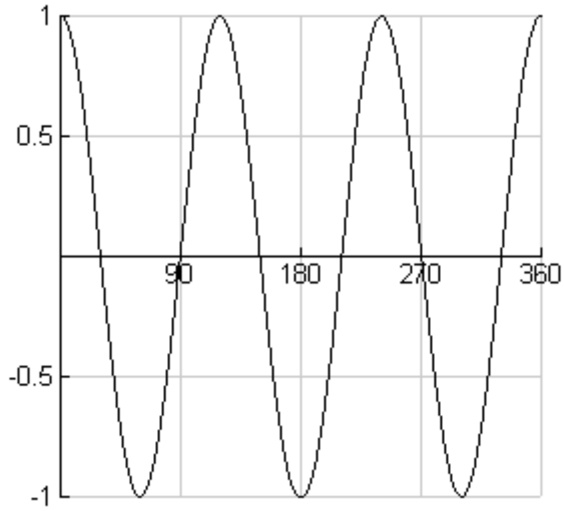
Adding a number to x has caused the curve to translate to the right.

*Multiplying the x number (**the period**)*

$y = \sin(3x)$



$y = \cos(3x)$

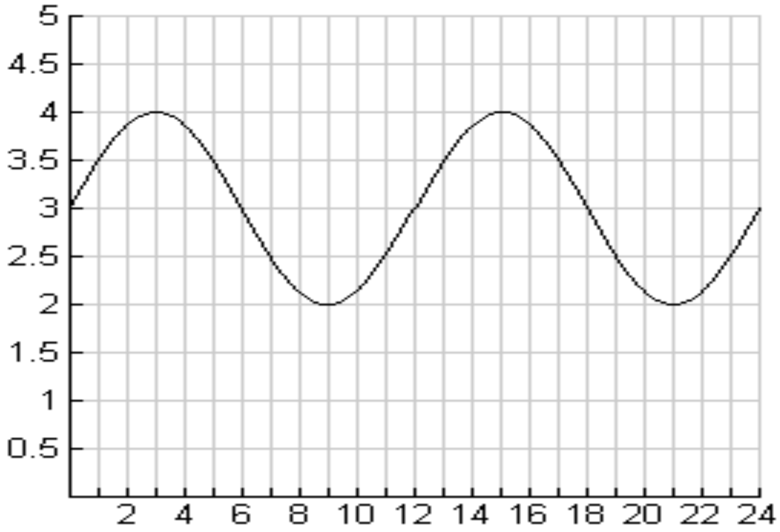


Multiplying the x number causes the curve to oscillate more. Each curve now has an **period** of 120°.

All of the above can be put together in modelling questions. Particular use should be made of the values of sin and cos that give 0 and 1. The worked example below shows all this information clearly.

Guided example

The height of the water in the harbour of Trumpton is tidal and modelled on a sine curve as shown in the diagram below.



The curve has the equation:

$$h = \sin(pt) + q$$

where t = hours after midnight and h = height of the water in metres.

(a) Use your graph to find the height of the water at 10 AM.

(b) A yacht can only safely enter and leave the harbour when there is more than 3 metres of water in the harbour.
Give the times of day when the ship can safely enter and leave the harbour.

(c) Find the values of p and q.

Answer (a)

Answer (a) and (b) can be found by simply reading the graph. At 10 AM $t = 10$, so there is approximately 2.2 metres of water in the harbour.

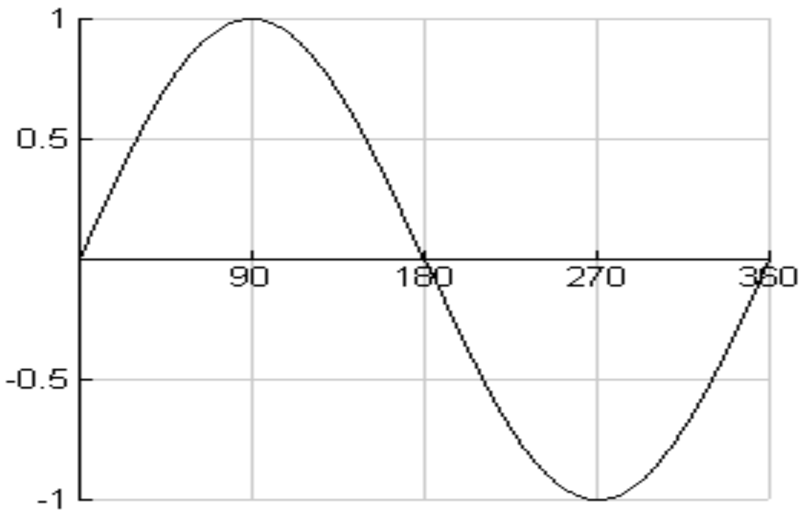
Answer (b)

By looking at the graph the height is above 3 for $0 \le t \le 6$ and $12 \le t \le 18$.
The corresponding times are:
Midnight to 6 AM and 12 PM and 6 PM.

Answer (c)
Since the curve has the equation $h = \sin(pt) + q$ we want to make use of $\sin 0^\circ = 0$. Looking at making $\sin(pt) = \sin(0)$, we can make $t = 0$. It does not matter what the value of p is by taking $t = 0$ the $\sin(pt)$ becomes 0.
When $t = 0$, $h = 3$, so we have the equation:

$$3 = \sin(0) + q$$
$$q = 3$$

Using this value and making use of sin 90 = 1. Remember the sine curve features. It may be useful to draw the $y = \sin x$ on your TI and compare it to the curve in the question.



When sin 90 = 1 this is the top of the graph. In the modelled graph the curve is at it's peak when $t = 3$.

We only need take the $\sin(pt) = 1$, so $pt = 90^\circ$ and we know that $t = 3$, so p is 30.

By drawing the curve of $y = \sin(30x) + 3$ on your TI you can check to see if it is the same as the curve given in the question (it is!).

$\Rightarrow$ $p = 30$ and $q = 3$

Trigonometric identities

You need to know the following trigonometric identities. They are all on the formulae sheet, but you should be familiar with using them to solve problems.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

*Note: If the question wants the **exact** number it must be given as a fraction.*

Remember: $\sin^2\theta = (\sin\theta)^2 \neq \sin(x)^2$.

These identities have been used below in the guided examples.

Guided example

a) Given that sin x = $\frac{1}{4}$, and that x is acute find the value of cos x.

b) Given that sin x = $\frac{5}{13}$, and that x is acute find the exact value of cos 2x.

c) Find the values of x that satisfy the equation,

$2\cos^2 x + \sin x = 2$ where x is an acute angle measured in degrees.

Answer (a)

As sin x = $\frac{1}{4}$ so $\sin^2 x = \frac{1}{16}$.

$\sin^2\theta + \cos^2\theta = 1$, so $\cos^2 x = \frac{15}{16}$.

$$\cos x = \frac{\sqrt{15}}{4}$$

Answer (b)

As sin x = $\frac{5}{13}$ so by using the method above we have $\cos^2 x = 1 - \frac{25}{169} = \frac{144}{169}$.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2x = \frac{144}{169} - \frac{125}{169} = \frac{109}{169}$$

Answer (c)

Use the identity $\sin^2\theta + \cos^2\theta = 1$ and some quadratics will solve this question.

$2\cos^2 x + \sin x = 2$

Substitute $\cos^2 x = 1 - \sin^2 x$ into the equation:

$$2(1-\sin^2 x) + \sin x = 2$$
$$2 - 2\sin^2 x + \sin x = 2$$
$$2\sin^2 x - \sin x = 0$$

Let $y = \sin x$ and substitute:

$2y^2 - y = 0$

$y(2y - 1) = 0$

$y = 0$ or $\frac{1}{2}$

So $\sin x = \frac{1}{2}$ or 0.

Using the calculator we get $x = 30$ or $x = 0$.

As x is acute we know the angle is 30°.