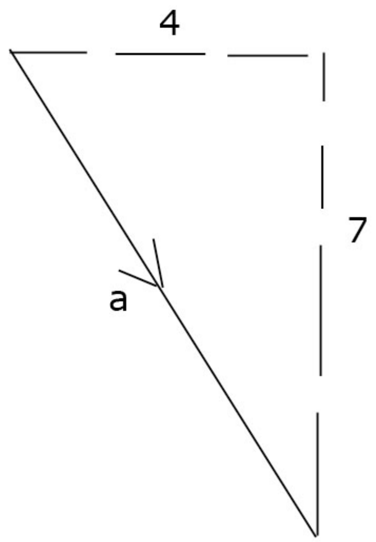
Basic vectors

Vectors are simply lines with magnitude (length/size) and direction. They are generally written in 2 dimensions with an $x \leftrightarrow$ component and a $y \updownarrow$ component.

These are written in the form **a** = $\begin{pmatrix} x \\ y \end{pmatrix}$ such as **a** = $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$. This vector can also be written in the form as:
4**i** – 7**j**.

This vector can be drawn as 4 units right and 7 units down, as shown below.



Magnitude

Vectors have a magnitude, which simply means the length. This can be calculated using Pythagoras' theorem as the *x* and *y* components will form a right-angled triangle. For the vector $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$, we can calculate the magnitude by $\sqrt{4^2 + (-7)^2}$ = 8.06.

Multiplication, addition and subtraction.
Vectors can be multiplied, added and subtracted.

Consider the vectors: **a** = $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ **b** = $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ **c** = $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$

2**a** = $\begin{pmatrix} 3\times 2 \\ 2\times 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ and **b** – **c** = $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$ - $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ = $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Unit vectors
A unit vector has the magnitude of 1 unit.
It can be evaluated easily by first finding the magnitude and then by dividing the *x* and *y* components by the magnitude.

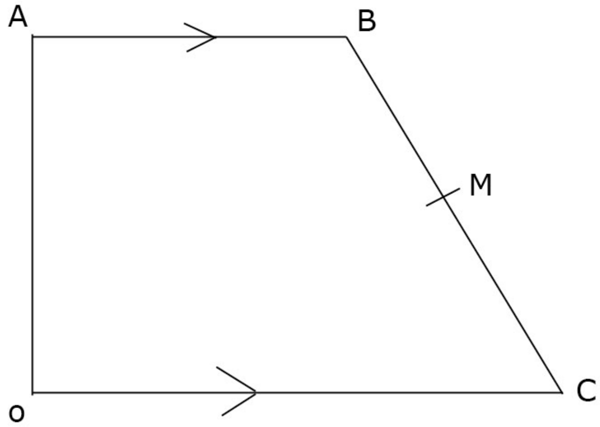
A simple example would be the vector $a = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

The magnitude of *a* = 5, so the unit vector of *a* would be $\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}$.

Vector geometry
Vectors are used in geometry to describe movement in shapes. With these types of question you will need to apply your knowledge of shapes. Often rectangles, parallelograms, trapeziums and various types of triangles are used. Frequently halfway points on lines are made and you are asked to find position vectors to point around a shape.

The position vector – this is the vector that starts at the origin, the point labelled *O* on the shape.

The best example of how to solve these questions can be seen by looking at the trapezium below.



$\overrightarrow{OA} = \mathbf{a}$ $\overrightarrow{OC} = \mathbf{c}$ *M* is at the point half way along *BC*, and *OC* = 3 *AB*

In terms of **a** and **c** we can find:

$\overrightarrow{AB} = \frac{1}{3}\overrightarrow{OC} = \frac{1}{3}\mathbf{c}$

OC is 3 *AB*

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \mathbf{a} + \frac{1}{3}\mathbf{c}$

With vectors you cannot always go in the direct route across the middle of the shape. You must use the vectors you are given or have previously found.

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AO} + \overrightarrow{OC}$

$= (-\frac{1}{3}\mathbf{c}) - \mathbf{a} + \mathbf{c}$

$= \frac{2}{3}\mathbf{c} - \mathbf{a}$

Note that if $\overrightarrow{OA} = \mathbf{a}$ then $\overrightarrow{AO} = -\mathbf{a}$.
This entire vector question is building on vectors already calculated.

The position vector of *M*

$\overrightarrow{OM} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC}$

$= (\mathbf{a} + \frac{1}{3}\mathbf{c}) + \frac{1}{2}(\frac{2}{3}\mathbf{c} - \mathbf{a})$

$= \frac{1}{2}\mathbf{a} + \frac{2}{3}\mathbf{c}$

Position vector tells us to start at *O*.
Although this may appear a little complicated at first, following it through step by step will just show that it is built on all our previous answers.

Scalar Product and the angle between two vectors

The scalar product between two vectors is defined as:

$$\mathbf{a.b} = |\mathbf{a}|\ |\mathbf{b}| \cos\theta$$

This refers to vectors **a** and **b** and the angle where they meet as θ.

a.b is the dot product which is the two *x*-components multiplied together and added with the two *y*-components.

$|\mathbf{a}|$ refers to the magnitude of the vector (see above).

Example
Find the obtuse angle between the vectors **a** and **b** given below.

$\mathbf{a} = 4\mathbf{i} + 6\mathbf{j}$ $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$

$\mathbf{a.b} = (4\text{x}2) + (6\text{x}-1) = 2$

$|\mathbf{a}| = \sqrt{4^2 + 6^2} = \sqrt{60}$

$|\mathbf{b}| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$

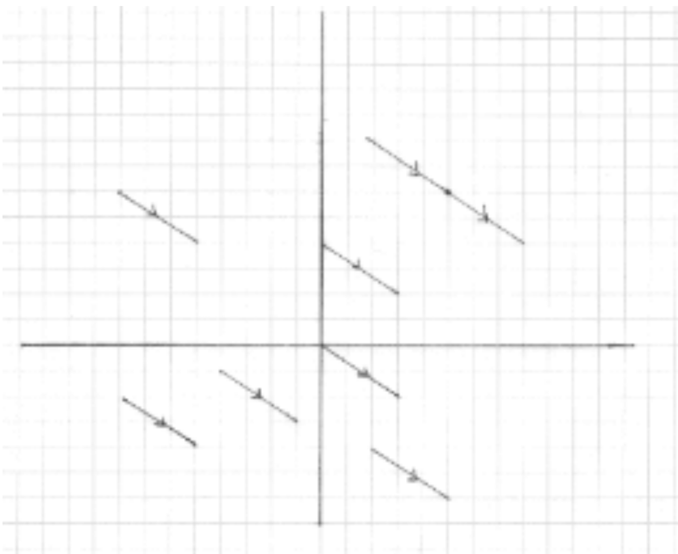
Rearranging the equation will give:

$\cos\theta = \dfrac{2}{\sqrt{60}\sqrt{5}}$ θ = 83.4° (from calculator).

As the question wants the obtuse angle we take θ from 180 to get the final answer of 96.6°.

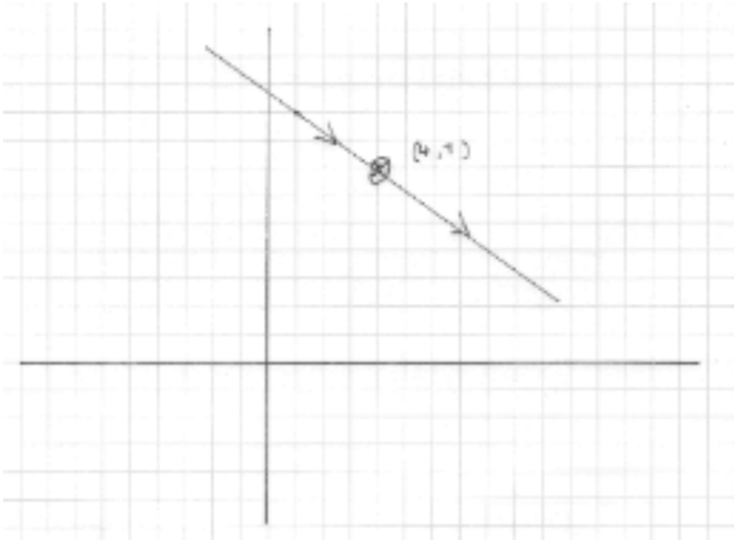
Vector straight line equations

Consider the vector 3**i** – 2**j**. This vector can be represented on a graph in the following ways:



Note all the vectors below go 3 across and down 2. The vector 3**i** – 2**j** can be multiplied by a scalar to get other vectors all going in the same direction as 3**i** – 2**j**, such as 24**i** – 16**j** or 9**i** – 6**j**.

Now consider a point on the graph such as (4,7). Consider the vector 3**i** – 2**j** that goes through this coordinate. It will look like this:



This straight line has a specific equation vector equation which is given as:

$$\mathbf{r} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

The first vector refers to the point which the vector goes through. The second vector is the vector 3**i** – 2**j** written in column form. The *t* is implying that it can be any multiple of this vector.

Guided example

A mathematician wishes to represent the map of Senegal with vectors. He considers the capital city, Dakar to be the origin. The town of Dagana has the vector **d** = 4**i** + 11**j** and the town of Maka has the vector equation **m = 15i** – 4**j**.

a) Draw a diagram of the information above showing the vectors **d** and **m**.

b) Write down in terms of **d** and **m** the vector that connects Dagana to Maka.

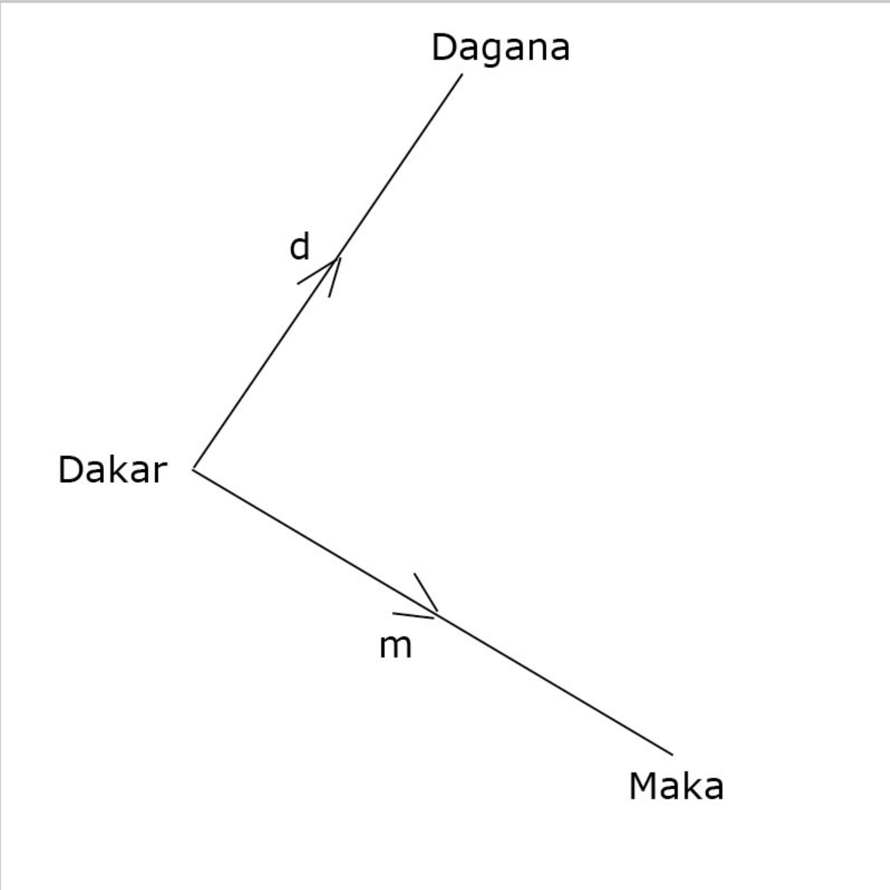
c) The town of Veliningara lies on the line between Dagana and Maka. It cuts the line in the ratio 2:3, being closer to Dagana.
Write down the vector that connects Dakar to Veliningara in terms of **d** and **m**.

d) Find the vector equation of the line connecting Dagana and Maka.

e) Find the angle made at Dakar between the vectors **d** and **m**.

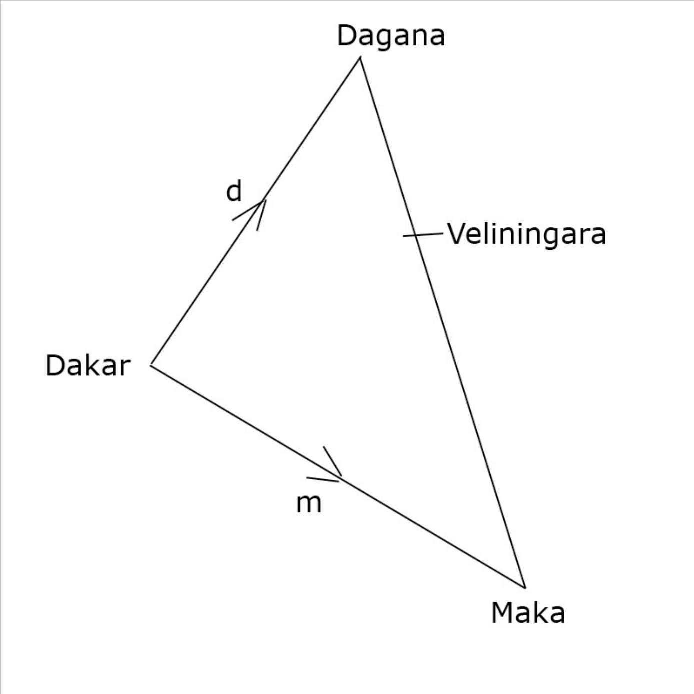
Answer

a)



b) m – d

c) The diagram below shows the location of V.



Now to get from D to V we can go first to D and then cut take $\frac{2}{5}$ of the vector from answer b) as follows:

d + $\frac{2}{5}$(m-d)

= $\frac{2}{5}$m + $\frac{3}{5}$d

d) The vector that connect D to M can be found by using m – d:

(15**i** – 4**j**) – (4**i** + 11**j**) = 11**i** **–** 15**j** or $\begin{pmatrix} 11 \\ -15 \end{pmatrix}$

Now take a point on the line. This can be either the coordinate at M or D, and add on a scalar product of the vector.

r = $\begin{pmatrix} 4 \\ 11 \end{pmatrix} + t\begin{pmatrix} 11 \\ -15 \end{pmatrix}$

e) We need to use the formula: **a.b = |a| |b| cos** θ

m.d = (60 – 44) = 26

|**m**| = $\sqrt{241}$ and |**d**| = $\sqrt{137}$

cos θ = $\frac{26}{\sqrt{241}\sqrt{137}}$

θ = 81.8°

3-d vectors

A third axis exists in space, *z*. Vectors can be shown in 3 dimensions using a third variable, *k*, such as:

a = 5i – 2j + 3k and **b** = 2i + 3j - k are both 3-d vectors.

The rules for 2-d vectors apply in 3 dimensions such,

$$|a| = \sqrt{5^2 + 2^2 + 3^2} = \sqrt{38}$$

$$|b| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$$

a.b = (5x2)+(-2x3)+(3x-1) = 1

The angle between the 2 3 dimensional vectors can also be obtained:

$$\cos\theta = \frac{1}{\sqrt{38}\sqrt{14}}$$

$$\theta = 87.5^\circ$$

Vector lines in 3-d

Vector lines in 3-d can be evaluated in the same manner as 2-d vectors.

Example:
Find the vector equation of the line that passes through point *p* with position vector $3i - 2j + 5k$, and the point *q* with position vector $i + 3j + 7k$.

The vector that connects *p* to *q* would be *Q*-*P*.

$$Q - P = -2i + 5j + 2k$$

So the vector equation can be written as $(3i - 2j + 5k) + t(-2i + 5j + 2k)$ or as $\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}$.